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VIBRATION OF CANTILEVERED LAMINATED COMPOSITE SHALLOW CONICAL SHELLS

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Abstract—This paper presents a new mathematical model to investigate the free vibration of laminated composite shallow conical shells in an effort towards accurate modelling of turbomachinery blades. The shells are composed of symmetrically or unsymmetrically laminated composites. This study focuses on E-glass/epoxy conical shells with four and eight-ply laminates, but the analysis can be easily extended to any composites with an arbitrary number of plies and fibre orientation. The energy integral expression is derived on the basis of the shallow shell theory incorporating variable chordwise surface curvature. A global computational approach based on the extremum energy principle is employed to derive the eigenvalue equation. A flexible, admissible global shape function is developed to account for the geometric boundary conditions. Previously unavailable frequency parameters and vibration mode shapes are presented. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Fibre reinforced composite laminates are increasingly used in many engineering disciplines for many products ranging from tiny to huge scales which require higher strength, more durability and less weight such as micro-mechanical components, aircraft and space vehicles. There are virtually unlimited ways of tailoring the mechanical properties of laminated composites to suit design requirements.

A vast literature is available on the vibration of shells. Most studies are confined to cylindrical or spherical shells with a constant radius of curvature (Cheung and Cheung, 1972; Fan and Cheung, 1983; Cheung *et al.*, 1989; Li *et al.*, 1990). Only a limited number of references are available for shells of other shapes, such as the cambered helicoidal shells by Walker (1978) and the conical shells by Srinivasan and Krishnan (1987) and Srinivasan and Hosur (1989). A review of the vibration of conical shells has been given by Chang (1981).

Studies of laminated composite shells have mainly focused on closed cylindrical and spherical shells, as reviewed by Kapania (1989) and Mirza (1991), because of their practical importance in the aerospace, petrochemical and nuclear industries. Mirza (1991) surveyed recent research on the vibration of laminated shells, assessing works published on free, forced vibration and dynamic response of layered shells. This review covers studies of analytical techniques, finite element analyses and experimental methods. Non-linear and large deformation effects were also presented.

Turbomachinery blades are conventionally modelled as beams (Carnegie, 1959), plates (Liew and Lim, 1994a; Lim and Liew, 1995a) and cylindrical shallow shells (Liew and

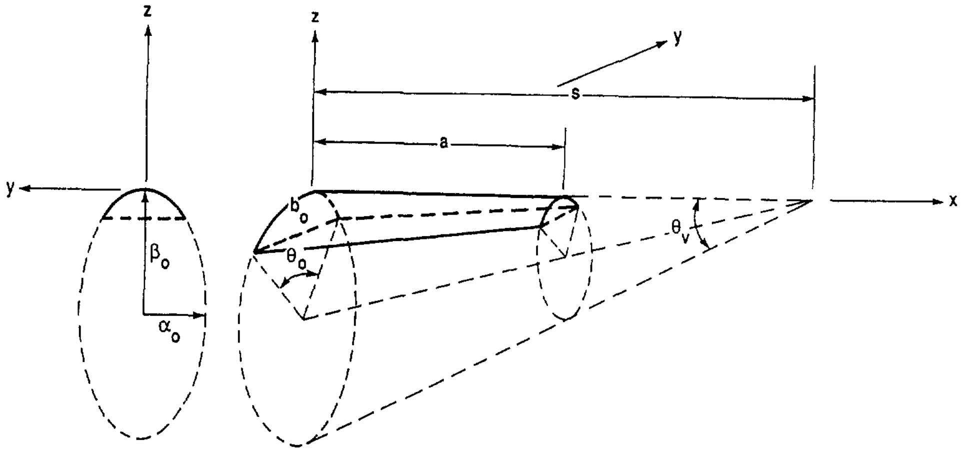

Fig. 1. Geometry of a laminated CFFF shallow conical shell.

Lim, 1994b). In Liew and Lim (1994a) and Lim and Liew (1995a), the vibration of a blade is modelled as a pretwisted plate with a trapezoidal planform. In another effort by Liew and Lim (1994b), the vibration of cylindrical shells with generally varying thickness has been investigated. One major deficiency of the cylindrical shallow shell model is the constant chordwise curvature. An actual turbomachinery blade should feature a shallow shell with both non-uniform planform and variable chordwise curvature. Thus an open conical shell model is more appropriate. Vibration of open conical shells has been reported by Lim and Liew (1995b) for isotropic, untwisted shells with uniform thickness.

The design of a turbomachinery blade is seldom confined to isotropic materials. Very frequently composite materials are preferred due to the advantages in strength, durability and weight. To the authors' knowledge, the vibration of laminated composite turbomachinery blades has not been undertaken using a shallow conical shell model. To fill this apparent void, a laminate-composite model with theory and solution methods has been developed to investigate the effects of symmetric and unsymmetric lamination on the vibration characteristics of cantilevered, open shallow conical shells. The Ritz energy principle is employed to formulate the governing eigenvalue equation and the kinematically orientated admissible *pb*-2 shape functions, developed by Liew and Lim and their associates (Liew and Lim, 1994a,b; Lim and Liew, 1995a,b), are further extended to laminated composite shells with multiple plies. Previously unavailable results covering a wide range of shell geometric and laminate configurations are presented for design applications and future comparison.

2. THEORETICAL CONSIDERATIONS

2.1. *Problem definition*

Consider a homogeneous, isotropic, thin shallow conical shell with midsurface length a, reference width b_0, thickness h, cone length s, vertex angle θ_v, base subtended angle θ_0 as illustrated in Fig. 1. The cone base can be assumed as elliptical with minor and major radii α_0 and β_0 since the shell is shallow. The radius of curvature in the chordwise direction $R_y(x, y)$ is a parameter varying in the x- and y-directions. The variation of $R_y(x, y)$ in the x-direction is linear. There is no curvature along the spanwise direction ($R_x = \infty$). This cantilevered shell is clamped along $x = 0$.

The midsurface geometry of a shallow conical shell with a trapezoidal planform is rather complex. From Fig. 1, the reference major radius and the major and minor radii β and α at any vertical cross-section is

$$\beta_0 = s\tan\frac{\theta_v}{2} \tag{1a}$$

$$\beta = \tan\frac{\theta_v}{2}(s-x) \tag{1b}$$

$$\alpha = \frac{b\beta \tan\dfrac{\theta_0}{2}}{\sqrt{4\beta^2 \tan^2\dfrac{\theta_0}{2} - b^2}} \tag{1c}$$

where

$$b = b_0\left[1 - \left(\frac{x}{s}\right)\right] \tag{2a}$$

and

$$b_0 = 2s \sin\frac{\theta_v}{2}\sqrt{\frac{\tan^2\theta_0/2}{\cos^2\theta_v/2 + \tan^2\theta_0/2}}. \tag{2b}$$

The equation of an ellipse at any perpendicular cross-section is

$$\left(\frac{y}{\alpha}\right)^2 + \left(\frac{z+\beta}{\beta}\right)^2 = 1. \tag{3}$$

Taking the first and second derivatives of z with respect to y and making use of the definition $k = (\mathrm{d}^2 z/\mathrm{d}y^2)/[1 + (\mathrm{d}z/\mathrm{d}y)^2]^{3/2}$, the chordwise radius of curvature is

$$R_y(x, y) = \left|\frac{1}{k}\right| = \alpha^2\beta^2\left[\frac{1}{\beta^2} + \frac{y^2}{\alpha^2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\right]^{3/2}. \tag{4}$$

2.2. *Mathematical formulation*

The total strain energy $\mathscr{U}$ of a laminated conical shell can be expressed as:

$$\mathscr{U} = \mathscr{U}_{\text{or}} + \mathscr{U}_{\text{es}} + \mathscr{U}_{\text{bs}} + \mathscr{U}_{\text{bt}} \tag{5}$$

where the strain energy of uncoupled orthotropic characteristics of the material (i.e. A_{11}, A_{12}, A_{22}, A_{66}, D_{11}, D_{12}, D_{22} and D_{66}) is

$$\begin{aligned}\mathscr{U}_{\text{or}} = \frac{1}{2}\int\!\!\int_A \Bigg\{ & A_{11}\left(\frac{\partial U}{\partial x}\right)^2 + A_{22}\left(\frac{\partial V}{\partial y}\right)^2 + A_{66}\left(\frac{\partial U}{\partial y}\right)^2 + A_{66}\left(\frac{\partial V}{\partial x}\right)^2 + 2A_{12}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}\right) \\ & + \frac{2A_{12}}{R_y(x,y)}\frac{\partial U}{\partial x}W + \frac{2A_{22}}{R_y(x,y)}\frac{\partial V}{\partial y}W + \frac{A_{22}}{R_y^2(x,y)}W^2 + 2A_{66}\left(\frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right) \\ & + D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial y^2}\right) + D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\Bigg\}\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{6a}$$

strain energy of extension-shearing coupling (i.e. A_{16} and A_{26}) is

$$\mathscr{U}_{es} = \iint_A \left\{ A_{16}\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} + A_{16}\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + A_{26}\frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + A_{26}\frac{\partial V}{\partial x}\frac{\partial V}{\partial y} + \frac{A_{26}}{R_y(x,y)}\frac{\partial U}{\partial y}W + \frac{A_{26}}{R_y(x,y)}\frac{\partial V}{\partial x}W \right\} \mathrm{d}x\,\mathrm{d}y \quad (6b)$$

strain energy of bending–stretching coupling (i.e. B_{ij}, $i,j = 1,\ldots,6$) is

$$\begin{aligned}\mathscr{U}_{bs} = -\iint_A \Bigg\{ & B_{11}\frac{\partial U}{\partial x}\frac{\partial^2 W}{\partial x^2} + B_{12}\frac{\partial U}{\partial x}\frac{\partial^2 W}{\partial y^2} + B_{16}\frac{\partial U}{\partial y}\frac{\partial^2 W}{\partial x^2} + 2B_{16}\frac{\partial U}{\partial x}\frac{\partial^2 W}{\partial x\,\partial y} \\ & + B_{26}\frac{\partial U}{\partial y}\frac{\partial^2 W}{\partial y^2} + 2B_{66}\frac{\partial U}{\partial y}\frac{\partial^2 W}{\partial x\,\partial y} + B_{12}\frac{\partial V}{\partial y}\frac{\partial^2 W}{\partial x^2} + B_{22}\frac{\partial V}{\partial y}\frac{\partial^2 W}{\partial y^2} \\ & + B_{16}\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x^2} + 2B_{26}\frac{\partial V}{\partial y}\frac{\partial^2 W}{\partial x\,\partial y} + B_{26}\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial y^2} + 2B_{66}\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\,\partial y} \\ & + \frac{B_{12}}{R_y(x,y)}W\frac{\partial^2 W}{\partial x^2} + \frac{B_{22}}{R_y(x,y)}W\frac{\partial^2 W}{\partial y^2} + \frac{2B_{26}}{R_y(x,y)}W\frac{\partial^2 W}{\partial x\,\partial y} \Bigg\} \mathrm{d}x\,\mathrm{d}y \end{aligned} \quad (6c)$$

and strain energy of bending–twisting coupling (i.e. D_{16} and D_{26}) is

$$\mathscr{U}_{bt} = 2\iint_A \left\{ D_{16}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial x\,\partial y} + D_{26}\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x\,\partial y} \right\} \mathrm{d}x\,\mathrm{d}y \quad (6d)$$

where the integration is on the planform area A of the shallow conical shell. The laminate stiffness coefficients A_{ij}, B_{ij} and D_{ij} for the k-ply are (Vinson and Sierakowski, 1986).

$$A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k - h_{k-1}), \quad (i,j = 1,2,6) \quad (7a)$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2), \quad (i,j = 1,2,6) \quad (7b)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3), \quad (i,j = 1,2,6) \quad (7c)$$

in which $\bar{Q}_{ij}$ $(i,j = 1,2,6)$ is the transformed stiffness depending on the ply stiffness constant Q_{ij} $(i,j = 1,2,6)$ and fibre orientation angle. The bending–stretching coupling vanishes ($B_{ij} = 0$) for symmetrically laminated shells.

The kinetic energy for free vibration is given by

$$\mathscr{T} = \frac{\rho h \omega^2}{2}\iint_A (U^2 + V^2 + W^2)\,\mathrm{d}x\,\mathrm{d}y \quad (8)$$

where ρ is the mass density per unit volume and ω is the rotational frequency.

The in-plane and transverse displacement amplitude functions can be approximated by a series of orthogonally generated two-dimensional polynomials

$$U(\xi,\eta) = \sum_{i=1}^{m} C_i^u \phi_i^u(\xi,\eta) \tag{9a}$$

$$V(\xi,\eta) = \sum_{i=1}^{m} C_i^v \phi_i^v(\xi,\eta) \tag{9b}$$

$$W(\xi,\eta) = \sum_{i=1}^{m} C_i^w \phi_i^w(\xi,\eta) \tag{9c}$$

where C_i^u, C_i^v and C_i^w are the unknown coefficients and ϕ_i^u, ϕ_i^v and ϕ_i^w are the corresponding admissible *pb*-2 shape functions in terms of a non-dimensional coordinate system defined as

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b_0} \tag{10a,b}$$

in which a and b_0 are the span and width of the shell as shown in Fig. 1.

The maximum strain energy $\mathscr{U}_{\max}$ and the maximum kinetic energy $\mathscr{T}_{\max}$ in a vibratory cycle occur at maximum and zero displacements, respectively. Following the Ritz extremum energy principle, the energy functional

$$\mathscr{F} = \mathscr{U}_{\max} - \mathscr{T}_{\max} \tag{11}$$

is minimized with respect to the unknown coefficients to obtain the following eigenvalue equation:

$$(\mathbf{K} - \Lambda^2 \mathbf{M})\{\mathbf{C}\} = \{\mathbf{0}\} \tag{12}$$

where **K** and **M** are the stiffness and mass matrices expressed as:

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}^{uu} & \mathbf{k}^{uv} & \mathbf{k}^{uw} \\ & \mathbf{k}^{vv} & \mathbf{k}^{vw} \\ \text{sym} & & \mathbf{k}^{ww} \end{bmatrix} \tag{13a}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}^{uu} & [0] & [0] \\ & \mathbf{m}^{vv} & [0] \\ \text{sym} & & \mathbf{m}^{ww} \end{bmatrix} \tag{13b}$$

the vector of unknown coefficients is

$$\{\mathbf{C}\} = \begin{Bmatrix} \{\mathbf{c}^u\} \\ \{\mathbf{c}^v\} \\ \{\mathbf{c}^w\} \end{Bmatrix} \tag{13c}$$

and the non-dimensional frequency parameter is

$$\Lambda = \frac{b_0}{h}\sqrt{12(1-\nu_{12}\nu_{21})}\,\lambda \tag{14a}$$

$$\lambda = \omega a \sqrt{\frac{\rho}{E_{11}}} \tag{14b}$$

where ρ is the density per unit volume, E_{11} is the Young's modulus along the fibre, and ν_{12} and ν_{21} are the Poisson's ratios. The elements in the stiffness and mass matrices (Lim and Liew, 1995b) are given in the Appendix.

The admissible *pb*-2 shape functions introduced in eqns (9a–c) consist of the product of terms of a two-dimensional orthogonally generated polynomial (*p*-2) and appropriate basic functions (b), i.e.

$$\phi_i^\alpha(\xi,\eta) = f_i(\xi,\eta)\phi_b^\alpha - \sum_{j=1}^{i-1} \Xi_{ij}^\alpha \phi_j^\alpha \tag{15}$$

where

$$\Xi_{ij}^\alpha = \frac{{}_1\Delta_{ij}^\alpha}{{}_2\Delta_j^\alpha} \tag{16a}$$

$${}_1\Delta_{ij}^\alpha = \iint_A f_i(\xi,\eta)\phi_b^\alpha \phi_j^\alpha \,\mathrm{d}\xi\,\mathrm{d}\eta \tag{16b}$$

$${}_2\Delta_j^\alpha = \iint_A (\phi_j^\alpha)^2 \,\mathrm{d}\xi\,\mathrm{d}\eta \tag{16c}$$

in which $\alpha = u, v$ or w and

$$f_i(\xi,\eta) = 1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \ldots, \tag{16d}$$

form a complete set of *p*-2 functions. Note that α here is only a dummy variable which is not related to the minor radius in Fig. 1. The basic functions are ϕ_b^α ($\alpha = u, v$ or w) defined as the products of the equations of the continuous piecewise boundary geometries of the shell planform each of which is raised to an appropriate basic power that corresponds to its geometric boundary condition.

In this study, the basic functions for the cantilevered conical shallow shell are

$$\phi_b^u = \xi, \quad \phi_b^v = \xi, \quad \phi_b^w = \xi^2 \tag{17a,b,c}$$

which satisfy the geometric boundary conditions at the clamped edge $\xi = 0$.

3. EXAMPLES AND DISCUSSION

Some numerical examples including convergence, comparison studies and vibration mode shapes are presented in this section. The composite considered in E-glass/epoxy with the following material properties:

E-glass/epoxy (E/E)

$$E_{11} = 60.7\,\text{GPa}$$
$$E_{22} = 24.8\,\text{GPa}$$
$$G_{12} = 12.0\,\text{GPa}$$
$$\nu_{12} = 0.23.$$

The shells are clamped along $\xi = 0$ and free on the other edges, denoted as CFFF.

Table 1. Convergence of $\lambda = \omega a \sqrt{\rho/E_{11}}$ for a CFFF E-glass/epoxy shallow conical shells with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$ and $\theta_f = 30°$

Lamination	θ_0	p: u v w	Mode frequencies 1	2	3	4
$[(-\theta_f, \theta_f)_2]_{sym}$	10°	11 11 11	0.015548	0.030968	0.078872	0.10133
		11 11 13	0.015547	0.030964	0.078865	0.10132
		11 11 15	0.015546	0.030962	0.078861	0.10131
		11 13 15	0.015545	0.030961	0.078856	0.10131
		11 15 15	0.015544	0.030961	0.078854	0.10131
		13 15 15	0.015543	0.030961	0.078851	0.10131
		15 15 15	0.015543	0.030961	0.078849	0.10131
	20°	11 11 11	0.027338	0.033318	0.10989	0.11458
		11 11 13	0.027337	0.033314	0.10988	0.11457
		11 11 15	0.027337	0.033312	0.10987	0.11456
		11 13 15	0.027333	0.033311	0.10987	0.11455
		11 15 15	0.027331	0.033310	0.10987	0.11455
		13 15 15	0.027330	0.033310	0.10987	0.11455
		15 15 15	0.027329	0.033310	0.10987	0.11455
	30°	11 11 11	0.035957	0.040272	0.12082	0.13010
		11 11 13	0.035953	0.040271	0.12080	0.13009
		11 11 15	0.035952	0.040270	0.12080	0.13008
		11 13 15	0.035950	0.040265	0.12079	0.13008
		11 15 15	0.035949	0.040262	0.12079	0.13008
		13 15 15	0.035949	0.040260	0.12079	0.13008
		15 15 15	0.035949	0.040259	0.12079	0.13008
$[(-\theta_f, \theta_f)_4]_{sym}$	10°	11 11 11	0.015629	0.031381	0.079268	0.10422
		11 11 13	0.015628	0.031377	0.079260	0.10421
		11 11 15	0.015627	0.031374	0.079255	0.10420
		11 13 15	0.015625	0.031373	0.079250	0.10419
		11 15 15	0.015625	0.031373	0.079248	0.10419
		13 15 15	0.015624	0.031373	0.079245	0.10419
		15 15 15	0.015624	0.031373	0.079243	0.10419
	20°	11 11 11	0.027438	0.033600	0.11243	0.11484
		11 11 13	0.027436	0.033596	0.11241	0.11483
		11 11 15	0.027436	0.033593	0.11240	0.11482
		11 13 15	0.027432	0.033592	0.11240	0.11482
		11 15 15	0.027430	0.033592	0.11240	0.11482
		13 15 15	0.027429	0.033591	0.11239	0.11481
		15 15 15	0.027428	0.033591	0.11239	0.11481
	30°	11 11 11	0.036391	0.040190	0.12343	0.12945
		11 11 13	0.036387	0.040188	0.12342	0.12944
		11 11 15	0.036385	0.040188	0.12341	0.12944
		11 13 15	0.036383	0.040183	0.12340	0.12944
		11 15 15	0.036382	0.040180	0.12340	0.12943
		13 15 15	0.036382	0.040178	0.12340	0.12943
		15 15 15	0.036382	0.040177	0.12340	0.12943

3.1. *Convergence study*

The convergence of eigenvalues for a cantilevered conical shell with 4- and 8-ply symmetric laminations is presented in Table 1. The degree p of the admissible *pb*-2 shape functions for u, v and w increases from 11 to 15. The number of terms m for u, v and w as expressed in eqn (16d) is related to the power by $m = (p+1)(p+2)/2$. The total number of terms in the admissible *pb*-2 shape functions in Table 1 increases from (78+78+78) to (136+136+136) and the determinant size of the eigenvalue equation (12) increases from $(78+78+78) \times (78+78+78)$ to $(136+136+136) \times (136+136+136)$.

Excellent convergence of eigenvalues can be observed in Table 1. The eigenvalues converge downwards as the degree of *pb*-2 functions is increased. Increasing the number of terms in the *pb*-2 functions results in shape functions with the higher flexibility. The shell stiffness and vibration frequencies are thus lower. From Table 1, $p = 15$ for u, v and w is adequate to provide excellent convergent frequencies. Unless otherwise stated, all subsequent numerical results are computed using $p = 15$ or $m = 136$ for each of the displacements.

Table 2. Frequency parameters $\lambda = \omega a\sqrt{\rho/E_{11}}$ for a CFFF E-glass/epoxy shallow conical shells with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$ and $\theta_0 = 30°$

Lamination	θ_f	Mode frequencies 1	2	3	4
$[(-\theta_f, \theta_f)_2]_{sym}$	0	0.033727	0.044772	0.12099	0.12207
	15	0.034688	0.043762	0.12107	0.12473
	30	0.035949	0.040259	0.12079	0.13008
	45	0.034834	0.036298	0.11712	0.13574
	60	0.032148	0.033685	0.11017	0.13669
	75	0.030752	0.031168	0.10277	0.13404
	90	0.029683	0.030690	0.099392	0.13229
$[(-\theta_f, \theta_f)_2]_{unsym}$	0	0.033727	0.044772	0.12099	0.12207
	15	0.034638	0.043723	0.12223	0.12366
	30	0.036082	0.040105	0.12319	0.12826
	45	0.035393	0.036092	0.11994	0.13451
	60	0.032286	0.034065	0.11233	0.13649
	75	0.031013	0.031146	0.10354	0.13400
	90	0.029683	0.030690	0.099392	0.13229
$[(-\theta_f, \theta_f)_4]_{sym}$	0	0.033727	0.044772	0.12099	0.12207
	15	0.034792	0.043752	0.12223	0.12423
	30	0.036382	0.040177	0.12340	0.12943
	45	0.035304	0.036501	0.12026	0.13572
	60	0.032280	0.034270	0.11262	0.13718
	75	0.030985	0.031248	0.10367	0.13427
	90	0.029683	0.030690	0.099392	0.13229
$[(-\theta_f, \theta_f)_4]_{unsym}$	0	0.033727	0.044772	0.12099	0.12207
	15	0.034780	0.043742	0.12270	0.12378
	30	0.036418	0.040135	0.12417	0.12880
	45	0.035415	0.036473	0.12101	0.13536
	60	0.032302	0.034372	0.11316	0.13712
	75	0.031026	0.031266	0.10386	0.13425
	90	0.029683	0.030690	0.099392	0.13229

3.2. *Numerical results and mode shapes*

New results for the non-dimensional frequency parameters λ, defined in eqn (14b), of CFFF E-glass/epoxy conical shallow shells are presented in Table 2 and Figs 2–5. Four types of laminations are considered: the 4- and 8-ply symmetric and unsymmetric laminates are considered in Table 2 where the fibre angle θ_f ranges from 0 to 90°. The first four vibration frequencies are presented in an ascending order. Attention will be focused on the effects of these parameters on the fundamental vibration frequency.

It is observed in Table 2 that the number of plies and layer lamination generally have minor effects on the vibration frequencies. The frequencies for 4- and 8-ply, symmetric and unsymmetric laminations differ by about 2% or less. For $\theta_f = 0°$ and 90°, the frequencies are correspondingly identical because the fibres are aligned in the x- or y-directions, respectively, and the multiple-ply laminate is equivalent to a single-ply laminate. Increasing the number of plies for a laminate generally increases the fundamental λ slightly, but the effect is not very significant. Only results for 4-ply symmetric laminates are presented subsequently.

The effect of θ_f on frequency response for shells with a base subtended angle $\theta_0 = 10$, 20 and 30° are shown in Figs 2–4 while the effect of varying θ_0 is shown in Fig. 5. Comparison of finite element solutions for the fundamental frequency is also presented in Figs 2–4. The FE solutions were obtained using LUSAS (a finite element package) with 15×30 semiloof elements. Convergence of the FE solutions have been tested. As observed, the Ritz solutions presented in this analysis agree well with the FE solutions. The analysis using the Ritz method requires far less computational effort (10% or less) than the finite element method (FEM). Only the fundamental frequency of the FEM is presented because it takes, comparatively, much longer time to obtain converged higher mode frequencies.

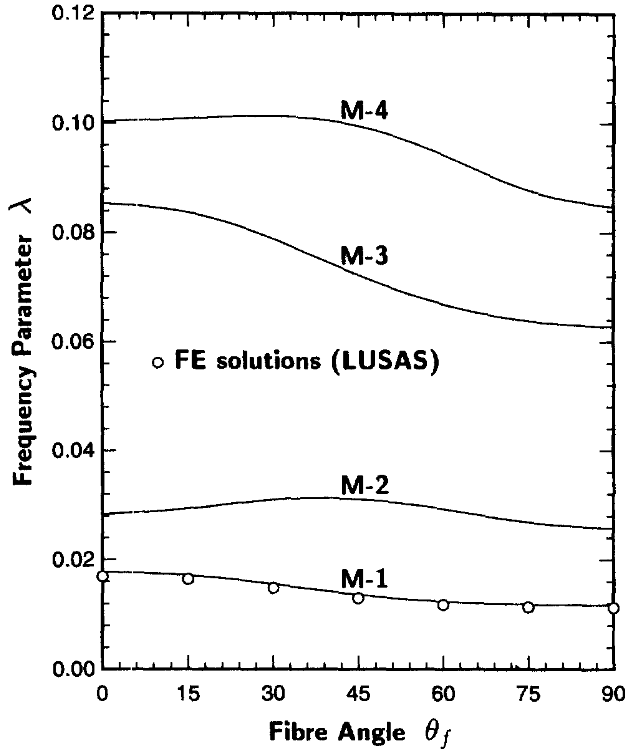

Fig. 2. Vibration frequency for a CFFF 4-ply E-glass/epoxy shallow conical shell with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$, $\theta_0 = 10°$ and stacking sequence $[(-\theta_f, \theta_f)_2]_{sym}$.

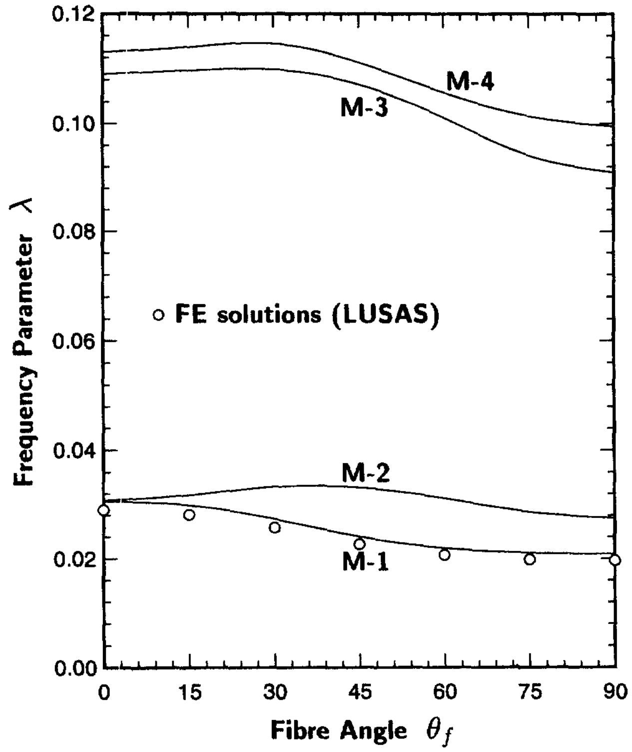

Fig. 3. Vibration frequency for a CFFF 4-ply E-glass/epoxy shallow conical shell with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$, $\theta_0 = 20°$ and stacking sequence $[(-\theta_f, \theta_f)_2]_{sym}$.

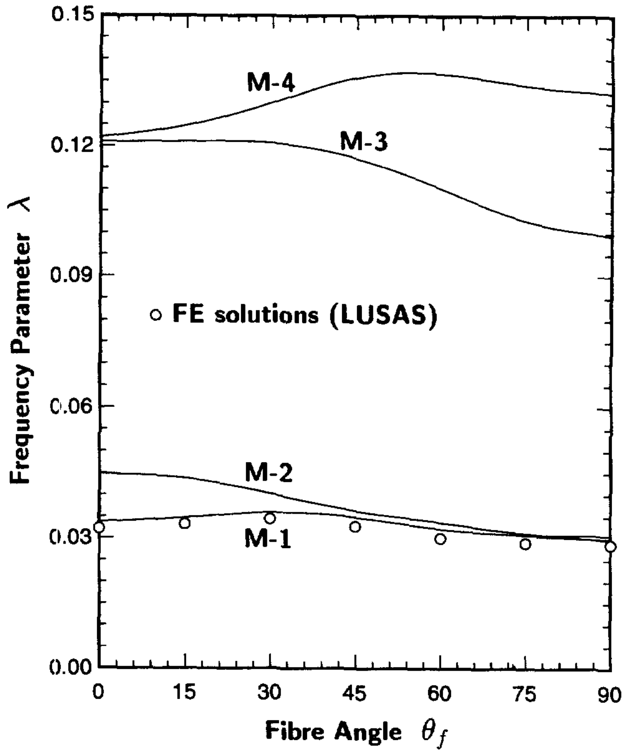

Fig. 4. Vibration frequency for a CFFF 4-ply E-glass/epoxy shallow conical shell with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$, $\theta_0 = 30°$ and stacking sequence $[(-\theta_f, \theta_f)_2]_{sym}$.

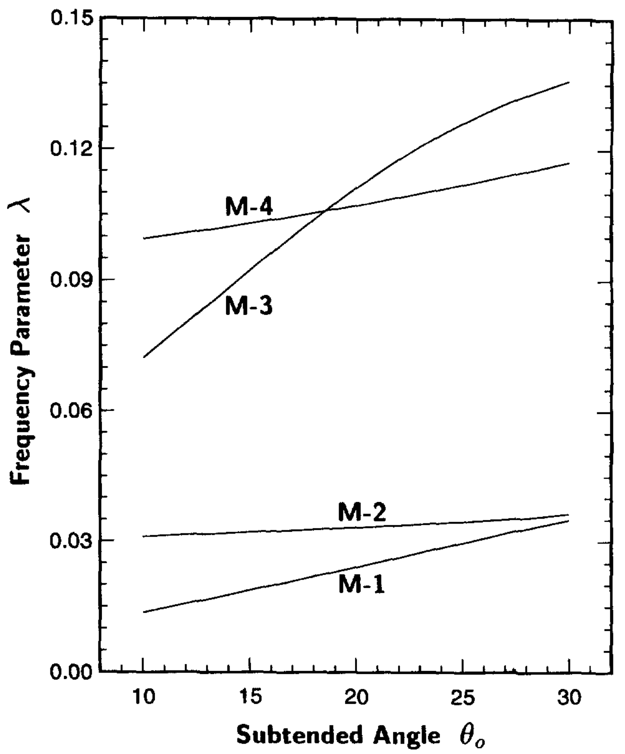

Fig. 5. Vibration frequency for a CFFF 4-ply E-glass/epoxy shallow conical shell with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$, $\theta_f = 45°$ and stacking sequence $[(-\theta_f, \theta_f)_2]_{sym}$.

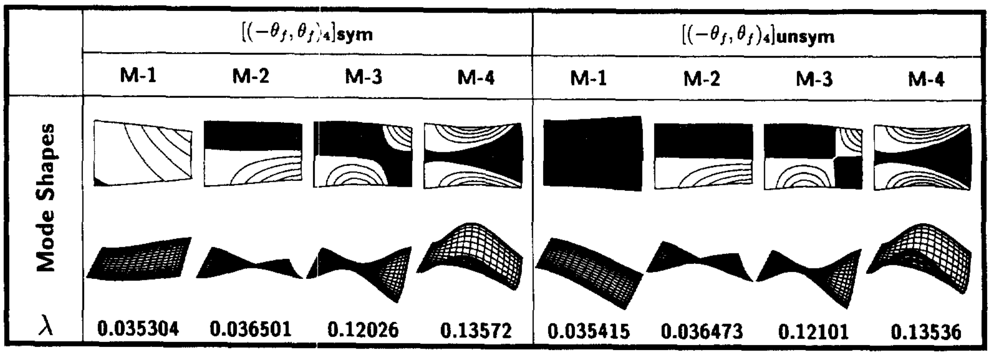

Fig. 6. Midsurface vibration mode shapes for a CFFF 8-ply E-glass/epoxy conical shallow shell with $a/h = 100.0$, $a/b_0 = 1.5$, $\theta_v = 15°$, $\theta_0 = 30°$, $\theta_f = 45°$.

In Fig. 5, an increase in θ_0 increases the first four mode frequencies. It is expected because an increase in θ_0 indicates an increase in the deepness of a shell and it has been well understood that the stiffness and frequencies of deeper shells is higher than shallower shells.

A set of midsurface vibration mode shapes is illustrated in Fig. 6. These correspond to the CFFF conical shell with 8-ply symmetric and unsymmetric laminations. To enhance the physical aspects of vibration, this mode shape figure illustrates the contour as well as three-dimensional midsurface displacement. The shaded and unshaded contour regions refer to regions with opposite vibration amplitude: one positive and the other negative. The lines of demarcation are the nodal lines with zero vibration amplitude. The fundamental mode in Fig. 6 is a spanwise bending mode while the second mode is a twisting mode. The third and fourth modes are coupled bending and twisting modes.

4. CONCLUSIONS

An analysis with computational solutions for the vibration of laminated shallow conical shells has been presented. Ritz energy principle has been employed to formulate the governing eigenvalue equation. The displacement amplitude functions have been approximated by geometrically admissible *pb*-2 shape functions. These functions ensure satisfaction of the geometric boundary conditions at the outset.

Previously unavailable, free vibration frequencies for shallow conical shells have been presented. The results cover various geometric and lamination parameters. Numerical solutions show that an increase in base subtended angle increases the vibration frequency. The effects of lamination symmetry and number of plies are minor, but laminates with more plies have slightly higher fundamental vibration frequency.

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APPENDIX

The elements in the stiffness and mass matrices (Lim and Liew, 1995b) are:

$$k_{ij}^{uu} = \frac{b_0^2 A_{11}}{D_0} \mathscr{I}_{uiuj}^{1010}(0) + \frac{ab_0 A_{16}}{D_0}[\mathscr{I}_{uiuj}^{0110}(0) + \mathscr{I}_{uiuj}^{1001}(0)] + \frac{a^2 A_{66}}{D_0}\mathscr{I}_{uiuj}^{0101}(0) \tag{A1}$$

$$k_{ij}^{uv} = \frac{ab_0 A_{12}}{D_0} \mathscr{I}_{uivj}^{1001}(0) + \frac{b_0^2 A_{16}}{D_0}\mathscr{I}_{uivj}^{1010}(0) + \frac{a^2 A_{26}}{D_0}\mathscr{I}_{uivj}^{0101}(0) + \frac{ab_0 A_{66}}{D_0}\mathscr{I}_{uivj}^{0110}(0) \tag{A2}$$

$$\begin{aligned} k_{ij}^{uw} = {} & \frac{ab_0^2 A_{12}}{\beta_0 D_0}\mathscr{I}_{uiwj}^{1000}(1) + \frac{a^2 b_0 A_{26}}{\beta_0 D_0}\mathscr{I}_{uiwj}^{0100}(1) - \frac{b_0^2 B_{11}}{aD_0}\mathscr{I}_{uiwj}^{1020}(0) - \frac{aB_{12}}{D_0}\mathscr{I}_{uiwj}^{1002}(0) \\ & - \frac{b_0 B_{16}}{D_0}[\mathscr{I}_{uiwj}^{0120}(0) + 2\mathscr{I}_{uiwj}^{1011}(0)] - \frac{a^2 B_{26}}{b_0 D_0}\mathscr{I}_{uiwj}^{0102}(0) - \frac{2aB_{66}}{D_0}\mathscr{I}_{uiwj}^{0111}(0) \end{aligned} \tag{A3}$$

$$k_{ij}^{vv} = \frac{a^2 A_{22}}{D_0}\mathscr{I}_{vivj}^{0101}(0) + \frac{ab_0 A_{26}}{D_0}[\mathscr{I}_{vivj}^{0110}(0) + \mathscr{I}_{vivj}^{1001}(0)] + \frac{b_0^2 A_{66}}{D_0}\mathscr{I}_{vivj}^{1010}(0) \tag{A4}$$

$$\begin{aligned} k_{ij}^{vw} = {} & \frac{a^2 b_0 A_{22}}{\beta_0 D_0}\mathscr{I}_{viwj}^{0100}(1) + \frac{ab_0^2 A_{26}}{\beta_0 D_0}\mathscr{I}_{viwj}^{1000}(1) - \frac{b_0 B_{12}}{D_0}\mathscr{I}_{viwj}^{0120}(0) - \frac{b_0^2 B_{16}}{aD_0}\mathscr{I}_{viwj}^{1020}(0) \\ & - \frac{a^2 B_{22}}{b_0 D_0}\mathscr{I}_{viwj}^{0102}(0) - \frac{aB_{26}}{D_0}[\mathscr{I}_{viwj}^{1002}(0) + 2\mathscr{I}_{viwj}^{0111}(0)] - \frac{2b_0 B_{66}}{D_0}\mathscr{I}_{viwj}^{1011}(0) \end{aligned} \tag{A5}$$

$$\begin{aligned} k_{ij}^{ww} = {} & \frac{a^2 b_0^2 A_{22}}{\beta_0^2 D_0}\mathscr{I}_{wiwj}^{0000}(2) - \frac{b_0^2 B_{12}}{\beta_0 D_0}[\mathscr{I}_{wiwj}^{0020}(1) + \mathscr{I}_{wiwj}^{2000}(1)] \\ & - \frac{a^2 B_{22}}{\beta_0 D_0}[\mathscr{I}_{wiwj}^{0002}(1) + \mathscr{I}_{wiwj}^{0200}(1)] - \frac{2ab_0 B_{26}}{\beta_0 D_0}[\mathscr{I}_{wiwj}^{0011}(1) + \mathscr{I}_{wiwj}^{1100}(1)] \\ & + \frac{b_0^2 D_{11}}{a^2 D_0}\mathscr{I}_{wiwj}^{2020}(0) + \frac{D_{12}}{D_0}[\mathscr{I}_{wiwj}^{0220}(0) + \mathscr{I}_{wiwj}^{2002}(0)] + \frac{2b_0 D_{16}}{aD_0}[\mathscr{I}_{wiwj}^{1120}(0) + \mathscr{I}_{wiwj}^{2011}(0)] + \frac{a^2 D_{22}}{b_0^2 D_0}\mathscr{I}_{wiwj}^{0202}(0) \\ & + \frac{2aD_{26}}{b_0 D_0}[\mathscr{I}_{wiwj}^{1102}(0) + \mathscr{I}_{wiwj}^{0211}(0)] + \frac{4D_{66}}{D_0}\mathscr{I}_{wiwj}^{1111}(0) \end{aligned} \tag{A6}$$

$$m_{ij}^{uu} = \mathscr{I}_{uiuj}^{0000}(0)\,; \quad m_{ij}^{vv} = \mathscr{I}_{vivj}^{0000}(0)\,; \quad m_{ij}^{ww} = \mathscr{I}_{wiwj}^{0000}(0) \tag{A7a–c}$$

where

$$\mathscr{I}_{uiuj}^{defg}(\gamma) = \iint_A \left[\frac{\beta_0}{R_y(\xi,\eta)}\right]^\gamma \frac{\partial^{d+e}\phi_i^u(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_j^u(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{A8}$$

$$\mathscr{I}_{uivj}^{defg}(\gamma) = \iint_A \left[\frac{\beta_0}{R_y(\xi,\eta)}\right]^\gamma \frac{\partial^{d+e}\phi_i^u(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_j^v(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{A9}$$

$$\mathscr{I}_{uiwj}^{defg}(\gamma) = \iint_A \left[\frac{\beta_0}{R_y(\xi,\eta)}\right]^\gamma \frac{\partial^{d+e}\phi_i^u(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\phi_j^w(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{A10}$$

$$\mathscr{I}_{vivj}^{defg}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_0}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^v(\xi,\eta)}{\partial\xi^d\partial\eta^e} \frac{\partial^{f+g}\phi_j^v(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{A11}$$

$$\mathscr{I}_{viwj}^{defg}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_0}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^v(\xi,\eta)}{\partial\xi^d\partial\eta^e} \frac{\partial^{f+g}\phi_j^w(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{A12}$$

$$\mathscr{I}_{wiwj}^{defg}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_0}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^w(\xi,\eta)}{\partial\xi^d\partial\eta^e} \frac{\partial^{f+g}\phi_j^w(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{A13}$$

and $i, j = 1, 2, \ldots, m$ where m depends on the degree of polynomial. The reference plate flexural rigidity is $D_0 = E_{11}h^3/12(1-\nu_{12}\nu_{21})$. The integration domain $\bar{A}$ is the normalized planform area of the shallow conical shell in accordance to eqns (10a, b).